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Sequential definitions of compactness

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Abstract

A subset F of a topological space is sequentially compact if any sequence $\mathbf{x} = (x_n)$ of points in F has a convergent subsequence whose limit is in F. We say that a subset F of a topological group X is G-sequentially compact if any sequence $\mathbf{x} = (x_n)$ of points in F has a convergent subsequence $\mathbf{y}$ such that $G(\mathbf{y}) \in F$ where G is an additive function from a subgroup of the group of all sequences of points in X. We investigate the impact of changing the definition of convergence of sequences on the structure of sequentially compactness of sets in the sense of G-sequential compactness. Sequential compactness is a special case of this generalization when $G = \lim$.

Keywords: Sequences; Series; Summability; Sequential compactness; Countable compactness

1. Introduction

One is often relieved to find that the standard neighbourhood definition of compactness for metric spaces can be replaced by a sequential definition of compactness. That many of the properties of compactness of sets can be easily derived using sequential arguments has also been, no doubt, a source of relief to the interested mathematics instructor.

We give some definitions and notation in the following. Throughout this work, $\mathbf{N}$ will denote the set of all positive integers. Although some of the definitions that follow make sense for an arbitrary topological group, that is why we prefer using neighborhoods instead of metrics. In this work, X will always denote a topological Hausdorff group, written additively, which satisfies the first axiom of countability. We will use bold-face letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n), \ldots$ of terms in X. $s(X)$, $c(X)$ and $C(X)$ denote the set of all X-valued sequences, the set of all X-valued convergent sequences and the set of all X-valued Cauchy sequences in X, respectively.

By a method of sequential convergence, or for short a method, we mean an additive function G defined on a subgroup of $s(X)$, denoted by $c_G(X)$, into X. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the group $c(X)$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$.

We discuss some special classes of methods of sequential convergence that have been studied in the literature including not only for real or complex number sequences but also for sequences of points in a topological Hausdorff

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group which satisfies the first axiom of countability. First of all, for real and complex number sequences, we note that the most important transformation class is the class of matrix methods. Consider an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence $A\mathbf{x}$ is defined as

$$A\mathbf{x} = \left(\sum_{k=1}^{\infty} a_{nk}x_k\right)_n$$

provided that each of the series converges. A sequence $\mathbf{x}$ is called A-convergent (or A-summable) to ℓ if $A\mathbf{x}$ exists and is convergent with

$$\lim A\mathbf{x} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}x_k = \ell.$$

Then ℓ is called the A-limit of $\mathbf{x}$. We have thus defined a method of sequential convergence, called a matrix method or a summability matrix method. The notion of regularity introduced above coincides with the classical notion of regularity for summability matrices. See [3] for an introduction to regular summability matrices.

The Hahn–Banach theorem can be used to define methods which are not generated from a regular summability matrix. Banach used this theorem to show that the limit functional can be extended from the convergent sequences to the bounded sequences while preserving linearity, positivity and translational invariance; these extensions have come to be known as Banach limits. If a bounded sequence is assigned to the same value ℓ by each Banach limit, the sequence is said to be almost convergent to ℓ. It is well known that a sequence $\mathbf{x} = (x_n)$ is almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+j} = \ell,$$

uniformly in j.

A sequence of additive functions f_m whose domain, for each m, is some subset of $s(X)$ and contains the set $C(X)$ and whose range is contained in X is said to constitute a limitation method. The intersection of the domains of the methods f_m is called the domain of the method (f_m) and let us denote (f_m) by F. Then, for any sequence $\mathbf{x} = (x_n)$, the sequence $F\mathbf{x}$ is defined as

$$F\mathbf{x} = (f_n(\mathbf{x}))_n.$$

A sequence $\mathbf{x}$ is F-convergent (or F-summable) to ℓ if $F\mathbf{x}$ is convergent with

$$\lim F\mathbf{x} = \lim_{n\to\infty} f_n(\mathbf{x}) = \ell.$$

Then ℓ is called the F-limit of $\mathbf{x}$. We have thus defined a method of sequential convergence, called a limitation method or a summability method in topological groups. The notion of regularity introduced above coincides with the classical notion of regularity for summability methods in topological groups. See [10] for an introduction to regular summability methods in topological groups.

Now we consider a class of methods that is unrelated to the preceding classes. Fast [12] introduced the definition of statistical convergence. Recall that for a subset M of $\mathbf{N}$ the asymptotic density of M, denoted by $\delta(M)$, is given by

$$\delta(M) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in M\}|,$$

if this limit exists, where $|\{k \le n : k \in M\}|$ denotes the cardinality of the set $\{k \le n : k \in M\}$. A sequence (x_n) is statistically convergent to ℓ if

$$\delta(\{n : |x_n - \ell| > \epsilon\}) = 0,$$

for every $\epsilon > 0$. In this case ℓ is called the statistical limit of $\mathbf{x}$. Statistically convergent sequences form a subspace of the linear space of all real-valued sequences and statistical limit is a linear functional on this space.

A sequence (x_k) of points in a topological group X, is called statistically convergent to an element ℓ of X if $\delta(M_U) = 0$ where $M_U = \{k : x_k - \ell \notin U\}$ for every neighborhood U of 0, i.e.

$$\delta(M_U) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : x_k - \ell \notin U\}| = 0,$$

for every neighbourhood U of 0. For an introduction to statistical convergence see [8,11–13,15–17]. A sequential method defined in this way is regular as every convergent sequence is statistically convergent with the same limit.

The notion of statistical convergence can be generalized to μ-statistical convergence by replacing the asymptotic density δ with an arbitrary density μ, that is, a finitely additive set function taking values in [0, 1] defined on a field of subsets of $\mathbf{N}$ with $\mu(\mathbf{N}) = 1$ such that if $|M| < \infty$ then $\mu(M)$=0 and if $M_1 \subset M_2$ with $\mu(M_2) = 0$ then $\mu(M_1) = 0$, cf. [5]. This notion covers several other variants of statistical convergence that have been considered in the literature.

A sequence (x_k) of points in a topological group is called lacunarily statistically convergent to an element ℓ of X if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : x_k - \ell \notin U\}| = 0,$$

for every neighborhood U of 0 where $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. For an introduction to lacunary statistical convergence for real or complex number sequences see [14]; and for sequences of points in topological groups, see [7,9]. The regularity of such methods depends on the lacunary sequence, namely, the method is regular if $\liminf_r q_r > 1$ where $q_r = \frac{k_r}{k_{r-1}}$. For an introduction to lacunary statistical convergence, see [7,9,14].

2. Definitions and notation

Now we give the definition of G-sequential closure of a subset of X.

Let $F \subset X$ and $\ell \in X$. Then ℓ is in the G-sequential closure of F (it is called G-hull of F in [6]) if there is a sequence $\mathbf{x} = (x_n)$ of points in F such that $G(\mathbf{x}) = \ell$. We denote G-sequential closure of a set F by $\overline{F}^G$. We say that a set is G-sequentially closed if it contains all of the points in its G-closure.

Now we can give the definition of G-sequential compactness of a subset of X.

Definition 1. A subset F of X is called G-sequentially compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in F there is a subsequence $\mathbf{y} = (x_{n_k})$ of $\mathbf{x}$ with $G(\mathbf{y}) \in F$.

For regular methods any sequentially compact subset of X is also G-sequentially compact and the converse is not always true.

In [6], a method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$ then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

For any regular subsequential method G, a subset F of X is G-sequentially compact if and only if it is sequentially compact in the ordinary sense.

Definition 2. A point ℓ is called a G-sequential accumulation point of F (or is in the G-sequential derived set) if there is a sequence $\mathbf{x} = (x_n)$ of points in $F \setminus \{\ell\}$ such that $G(\mathbf{x}) = \ell$.

Definition 3. A subset F of X is called G-sequentially countably compact if any infinite subset of F has at least one G-sequential accumulation point in F.

Definition 4. A function f is called G-sequentially continuous at $u \in X$ provided that whenever a sequence $\mathbf{x} = (x_n)$ of terms in X is G-convergent to u then the sequence $f(\mathbf{x}) = (f(x_n))$ is G-convergent to $f(u)$. For a subset D of X, f is called G-sequentially continuous on D if it is G-sequentially continuous at every $u \in D$, and is G-sequentially continuous if it is G-sequentially continuous on X (see also [1,2,4,18,19]).

3. Theorems and corollaries

First of all, we note that any finite subset of X is G-sequentially compact, the union of two G-sequentially compact subsets of X is G-sequentially compact and the intersection of any G-sequentially compact subsets of X is G-sequentially compact.

Theorem 1. *Any G-sequentially closed subset of a G-sequentially compact subset of X is G-sequentially compact.*

Proof. Let F be any G-sequentially compact subset of X and B be a G-sequentially closed subset of F. Take any sequence $\mathbf{x} = (x_n)$ of points in B. Then $\mathbf{x}$ is a sequence of points in F. Since F is G-sequentially compact, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ such that $G(\mathbf{y}) \in F$. The subsequence $\mathbf{y}$ is also a sequence of points in B. Since B is G-sequentially closed, $G(\mathbf{y}) \in B$. Thus $\mathbf{x}$ has a G-convergent subsequence, with $G(\mathbf{y}) \in B$, so B is G-sequentially compact. $\square$

Theorem 2. *Let G be a regular subsequential method. Any G-sequentially compact subset of X is G-sequentially closed.*

Proof. Let F be any G-sequentially compact subset of X. Take any $\ell \in \overline{F}$. Then there is a sequence $\mathbf{x} = (x_n)$ of points in F such that $G(\mathbf{x}) = \ell$. Since G is a subsequential method, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ such that $\lim_k x_{n_k} = \ell$. Since G is regular, $G(\mathbf{y}) = \ell$. By the G-sequential compactness of F, there is a subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ such that $G(\mathbf{z}) = \ell_1 \in F$. Since $\lim_k z_k = \ell$ and G is regular, $G(\mathbf{z}) = \ell$. Thus $\ell = \ell_1$ and hence $\ell \in F$. Thus F is G-sequentially closed. $\square$

Corollary 3. *Let G be a regular subsequential method. Then any G-sequentially compact subset of X is closed in the ordinary sense.*

Theorem 4. *Let G be a regular subsequential method. Then a subset of X is G-sequentially compact if and only if it is G-sequentially countably compact.*

Proof. Let F be any G-sequentially compact subset of X and B be an infinite subset of F. We can choose a sequence $\mathbf{x} = (x_n)$ of different points of B. G-sequential compactness of F implies that the sequence $\mathbf{x}$ has a convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ with $G(\mathbf{y}) = \ell$. Since G is a subsequential method, $\mathbf{y}$ has a convergent subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ with $\lim \mathbf{z} = \ell$. From the regularity of G, we obtain that ℓ is a G-sequential accumulation point of B. Thus F is G-sequentially countably compact. Now suppose that F is any G-sequentially countably compact subset of X. Let $\mathbf{x} = (x_n)$ be any sequence of points in F. Write $C = \{x_n : n \in \mathbf{N}\}$. If C is finite, then there is nothing to prove. If C is infinite, then C has a G-sequential accumulation point in F. Also each set $C_n = \{x_k : k \geq n\}$, for each positive integer n, has a G-sequential accumulation point in F. Then the intersection $\bigcap_{n=1}^{\infty} \overline{C_n}^{G}$ is not empty. So there is an element ℓ of F which belongs to the intersection. Since G is a regular subsequential method, $\ell \in \bigcap_{n=1}^{\infty} \overline{C_n}$. Then it is not difficult to construct a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ with $G(\mathbf{z}) \in F$. This completes the proof. $\square$

Corollary 5. *Let G be a regular subsequential method. Then a subset of X is G-sequentially compact if and only if it is sequentially countably compact in the ordinary sense.*

Corollary 6. *Let G be a regular subsequential method. Then a subset of X is G-sequentially compact if and only if it is countably compact in the ordinary sense.*

Theorem 7. *The G-sequential continuous image of any G-sequentially compact subset of X is G-sequentially compact.*

Proof. Let f be any G-sequentially continuous function on X and F be any G-sequentially compact subset of X. Take any sequence $\mathbf{y} = (y_n) = (f(x_n))$ of points in $f(F)$. Since F is G-sequentially compact, there exists a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x} = (x_n)$ with $G(\mathbf{z}) \in F$. Then the sequence $f(\mathbf{z}) = (f(z_k)) = (f(x_{n_k}))$ is a subsequence of the sequence $\mathbf{y}$. Since f is G-sequentially continuous, $G(f(\mathbf{z})) = f(u) \in f(F)$. Thus $f(F)$ is G-sequentially compact. $\square$

For regular subsequential methods we have much more.

Corollary 8. *Let G be a regular subsequential method. Then a G-sequentially continuous image of any sequentially compact subset of X is sequentially compact.*

Proof. The proof can be obtained by using techniques similar to that used for the preceding theorem and Corollary 5. □

Let us denote the set of all G-sequentially compact subsets of X by κ_C where G is a regular subsequential method. If we denote the set of all complements of elements of κ_C by τ_C then τ_C becomes a topology on X and this topology is coarser than the original topology of the topological group X, i.e. $\tau_C \subset \tau$ where τ is the original topology that makes X a topological group.

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